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Light-cone Wilson loops and the string/gauge correspondence

Yuri Makeenko

Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya 25, 117259 Moscow, Russia, and
The Niels Bohr Institute
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
E-mail: makeenko@itep.ru, makeenko@nbi.dk

Abstract: We investigate a Π-shape Wilson loop in $\mathcal{N}=4$ super Yang–Mills theory, which lies partially at the light-cone, and consider an associated open superstring in $AdS_5 \times S^5$. We discuss how this Wilson loop determines the anomalous dimensions of conformal operators with large Lorentz spin and present an explicit calculation in perturbation theory to order λ. We find the minimal surface in the supergravity approximation, that reproduces the Gubser, Klebanov and Polyakov prediction for the anomalous dimensions at large $\lambda = g^2_{\rm YM} N$, and discuss its quantum-mechanical interpretation.

Keywords: AdS-CFT and dS-CFT Correspondence, 1/N Expansion.

Contents

1. Introduction

There is a long-standing belief [1] that SU(N) Yang–Mills theory at large-N is equivalent to a free string, while the $1/N$-expansion corresponds to interactions of the string. A great recent progress along this line for $\mathcal{N}=4$ super Yang–Mills theory (SYM) is associated with the AdS/CFT correspondence [2], where the strong-coupling limit of SYM is described by supergravity in anti-de Sitter space $AdS_5 \times S^5$.

Among the most interesting predictions of the AdS/CFT correspondence for the strong-coupling limit of SYM, let us mention the calculation [3, 4] of anomalous dimensions of certain operators (see ref. [5] for a review) and that [6, 7] of the euclidean-space rectangular Wilson loop determining the interaction potential. The former is given by the spectrum of excitations in AdS space, while the latter is given by the minimal surface formed by the worldsheet of an open string whose ends lie at the loop in the boundary of $AdS_5 \times S^5$. The computations of the Wilson loops in the supergravity approximation were performed also for circular loops [8, 9] and loops with cusps [9]. The circular Wilson loop has then been

[1] See e.g. ref. [1] for an introduction and review of the old works on the string/gauge correspondence.

exactly calculated [10] in SYM to all orders in the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$. The result provided not only a beautiful test of the AdS/CFT correspondence at large λ but also a challenging prediction [11] for IIB superstring in the $AdS_5 \times S^5$ background.

Yet another remarkable test of the string/gauge correspondence concerns [12] a certain class of operators in SYM, whose anomalous dimensions can be exactly computed as a function of λ both in string theory and under some mild assumptions in SYM [13, 14]. The exact computation in string theory is possible because the anomalous dimensions of these BMN operators correspond to the spectrum of states with large angular momentum associated with rotation of an infinitely short closed string around the equator of S^5.

Rotating similarly a long closed folded string in AdS_5, a very interesting prediction concerning the strong-coupling limit of the anomalous dimensions of twist (= bare dimension minus Lorentz spin n) two operators has been obtained recently in ref. [15]:

$$\Delta - n = f(\lambda) \ln n \tag{1.1}$$

for large-n, where

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \mathcal{O}\left((\sqrt{\lambda})^0\right) \tag{1.2}$$

and $\lambda = g_{\rm YM}^2 N$ — the 't Hooft coupling in SYM — is large. The correction $\mathcal{O}((\sqrt{\lambda})^0)$ has been calculated [16] and it has been shown [17] very recently how the GKP result can be reproduced via a minimal surface of an open string spanned in the boundary by the loop with a cusp.

Equations (1.1) and (1.2) were derived [15] ignoring the S^5 part of $AdS_5 \times S^5$, which is responsible for supersymmetry, and possess the features expected for the anomalous dimension in ordinary (nonsupersymmetric) Yang–Mills theory. There are arguments for this result to be valid in ordinary Yang–Mills theory as well. This would lead us to very interesting predictions for the strong-coupling limit of QCD!

The goal of this paper is to further study the string/gauge correspondence considering a Wilson loop with cusps, which partially lies at the light cone, on the SYM side and minimal surfaces in AdS space associated with an open string ending at the loop in the boundary. We extend the results of ref. [18] to the SYM case and show how the vacuum expectation value of this Wilson loop determines the anomalous dimensions of the twist-two operators in SYM perturbation theory. We find an appropriate minimal surface in AdS space and show that it reproduces the GKP result (1.1), (1.2).

This paper is organized as follows. After a brief excursion to QCD, we define in section 2 the light-cone Wilson loop in SYM, discuss its properties and demonstrate in perturbation theory how it gives the anomalous dimensions of the twist-two operators. An explicit calculation is presented to order λ. In section 3 we find a solution for the minimal surface associated with the worldsheet of an open string in the AdS background, the ends of which lie at the loop in the boundary, and demonstrate how it reproduces eqs. (1.1) and (1.2). We discuss some unusual properties of this solution and its quantum-mechanical interpretation as tunneling in an analogous mechanical problem.

2. Light-cone Wilson loop in SYM

2.1 The set up

The relation between the anomalous dimensions of hadronic operators $\mathcal{O}_n$ with the Lorentz spin n in QCD and the renormalization of open Wilson loops with quarks at the ends is well-known [19, 20]. A crucial role in this correspondence is played [21, 22, 23] by conformal operators which are multiplicatively renormalizable at one loop.

A convenient formulation of this approach was proposed by Korchemsky and Marchesini [18] who considered a Π-shape Wilson loop

$$\mathrm{U}(\Pi) = \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_\Pi \mathrm{d}x^\mu A_\mu} \tag{2.1}$$

with the ends at infinity, whose middle segment lies at the light cone. It is depicted in figure 1.

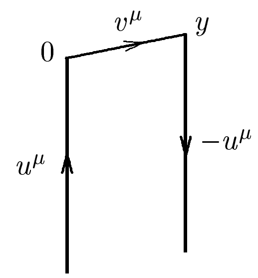

Figure 1: Π-shape Wilson loop. The segment $[0, y^\mu = v^\mu T]$ lies at the light cone. The loop is analytically given by eq. (2.2).

This Wilson loop can be parametrized by

$$x^\mu(t) = \begin{cases} u^\mu t & \text{for} \quad -\infty < t < 0 \\ v^\mu t & \text{for} \quad 0 \le t \le T \\ v^\mu T - u^\mu (t-T) & \text{for} \quad T < t < \infty\,, \end{cases} \tag{2.2}$$

where the unit vector u^μ is time-like ($u^2 = -1$) and the segment $[0, y^\mu = v^\mu T]$ lies at the light cone ($v^2 = 0$). Without loss of generality we can choose

$$u^\mu = (-1,0,0,0)\,, \qquad v^\mu = (1,1,0,0) \tag{2.3}$$

so that $y^\mu = (T,T,0,0)$. For the general case, we define

$$L = uv\,T\,. \tag{2.4}$$

The vacuum expectation value of such a Wilson loop

$$W(\Pi) \stackrel{\mathrm{def}}{=} \left\langle \frac{1}{N} \operatorname{tr} \mathrm{U}(\Pi) \right\rangle \tag{2.5}$$

is a function of the ratio

$$\rho = \frac{L}{\epsilon}\,, \tag{2.6}$$

where L is defined by eq. (2.4) and ϵ is an ultraviolet cutoff. It is the only dimensionless parameter which is present.

Though this Π-shape Wilson loop is *not* renormalizable owing to additional light-cone divergences, its logarithmic derivative

$$\Gamma(\Pi) \stackrel{\mathrm{def}}{=} -\epsilon \frac{\mathrm{d}}{\mathrm{d}\,\epsilon} \log W(\Pi) \tag{2.7}$$

is multiplicatively renormalizable when expressed via the renormalized coupling constant, so that $1/\epsilon$ in eq. (2.6) can be replaced for $\Gamma(\Pi)$ by the renormalization-group scale μ.

After an analytic continuation to imaginary ρ, the logarithmic derivative (2.7) determines the anomalous dimensions γ_n of the hadronic operators $\mathcal{O}_n$ with large-n by the formula [18]

$$\Gamma(\Pi)\big|_{\rho=-in} = \gamma_n \,. \tag{2.8}$$

Owing to the renormalizability of $\Gamma(\Pi)$, γ_n depends on n logarithmically

$$\gamma_n = f(\lambda) \log n \,, \tag{2.9}$$

where $f(\lambda)$ is known to a few lower orders of perturbation theory.

Equations (2.7) and (2.8) can be understood considering the operator

$$\mathcal{O}(C_{y0}) = \bar{\psi}(y)\,\mathrm{U}(C_{y0})\psi(0) \tag{2.10}$$

associated with the straight open Wilson loop C_{y0} with the (scalar or spinor) matter field ψ attached at the end points. Then

$$\left\langle \psi\left(\infty,\vec{0}\right)\mathcal{O}(C_{y0})\bar{\psi}\left(\infty,\vec{y}\right)\right\rangle \propto W(\Pi) \tag{2.11}$$

in the limit of the large mass of the matter field. In other words, the Π-shape Wilson loop is associated with the trajectory of a heavy particle which is at rest at the spacial point $\vec{0}$ for $-\infty < t < 0$, moves from $\vec{0}$ to $\vec{y}$ along the light cone during the time y_0 and then stays at rest at $\vec{y}$ for $y_0 \leq t < \infty$. It is clear from the analysis of perturbation-theory diagrams that the anomalous dimensions γ_n with large-n are correctly reproduced in the large-mass limit.

2.2 Extension to $\mathcal{N} = 4$ SYM

We shall now extend the results, reviewed in the previous subsection, to the $\mathcal{N} = 4$ SYM.

We use the known supersymmetric extension [6, 7, 9] of the Wilson loops in Minkowski space and define

$$\mathrm{U}(C) = \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}s\,(\dot{x}^\mu A_\mu + |\dot{x}|\theta^i\Phi_i)}\,, \tag{2.12}$$

where Φ_i are six scalar fields and θ^i is a unit vector in $\mathbb{R}^6$. We shall be interested in the case when C is the Π-shape loop, the parametrization of which is defined in eq. (2.2).

The difference between the Minkowski-space Wilson loops (2.12) and their euclidean-space counterparts, which have been extensively studied in the literature, is that the latter has an extra factor of i in front of Φ_i in the exponent. The Minkowski-space Wilson loop (2.12) is a BPS state, in particular, for an infinite straight line which lies inside the light cone, i.e. it can always be made parallel to the temporal axis by a Lorentz boost. For such loops

$$\left\langle \frac{1}{N}\,\mathrm{tr}\,\mathrm{U}\,(|)\right\rangle = 1 \tag{2.13}$$

in a full analogy with straight lines in euclidean space.

We shall be interested in anomalous dimensions of conformal operators built out of the fields in SYM which belong to the adjoint representation of SU(N). Consequently, the Wilson loops are to be taken in the adjoint representation:

$$\mathrm{tr}_{\mathrm{A}}\,\mathrm{U}(C) = |\,\mathrm{tr}\,\mathrm{U}(C)|^2 - 1\,. \tag{2.14}$$

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Owing to the large-N factorization, we have

$$W_{\rm A}(C) \stackrel{\rm def}{=} \left\langle \frac{1}{N^2-1} {\rm tr}_{\rm A}\, {\rm U}(C) \right\rangle \stackrel{N\to\infty}{=} W^2(C)\,, \tag{2.15}$$

where the right-hand side is defined by eqs. (2.5) and (2.12) for the fundamental representation. Thus, the anomalous dimensions of such adjoint operators are twice larger in the large-N limit than those in the fundamental representation.

2.3 Order λ of perturbation theory

The expectation value of the adjoint Wilson loop to the order λ is the same as in the U(1)-case and is given by

$$W_{\rm A}(\Pi) = 1 - \frac{\lambda}{2} \int\limits_{-\infty}^{+\infty} {\rm d}t_1 \int\limits_{-\infty}^{+\infty} {\rm d}t_2 \Big[\dot{x}^\mu(t_1)\dot{x}_\mu(t_2) + |\dot{x}(t_1)||\dot{x}(t_2)|\Big] D\left(x(t_1)-x(t_2)\right), \tag{2.16}$$

where the factor of 1/2 is related to the normalization of the SU(N) generators t^a.

The scalar propagator in d dimensions reads as

$$D(x) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \left(x^2\right)^{1-d/2}. \tag{2.17}$$

The relative sign of the two terms in square brackets in eq. (2.16) is minus for the temporal component since the gauge-field propagator in Minkowski space is

$$D_{\mu\nu}(x) = \eta_{\mu\nu} D(x)\,, \qquad \eta_{\mu\nu} = {\rm diag}\,(-\,+\,++) \tag{2.18}$$

in the Feynman gauge.

To regularize divergent integrals, we shall use either dimensional regularization when $0 < 4-d \ll 1$ or a smearing when $D(x)$ is substituted by

$$D_\epsilon(x) = \frac{1}{4\pi^2(x^2+\epsilon^2)}\,. \tag{2.19}$$

Though this smearing is not gauge invariant, it will be enough for our purposes since the $\mathcal{N}=4$ SYM has no charge renormalization in $d=4$ dimensions, so the only role of the regularization is to regularize the light-cone or cusp singularities of the expectation value of the Wilson loop.

Depending on which segment of the loop the points $x(t_1)$ and $x(t_2)$ belong to, the right-hand side of eq. (2.16) can be represented as the sum of the six diagrams in figure 2. The diagrams in figures 2a and f vanish quite similarly to the case of an infinite straight line (cf. eq. (2.13)), since these straight segments are time-like. The diagram in figure 2d vanishes since the ends of the propagators are at the light cone where $|\dot{x}|=0$. For the same reason the contribution of the diagrams in figures 2b and e is as in ref. [18]: these of the scalars vanish.

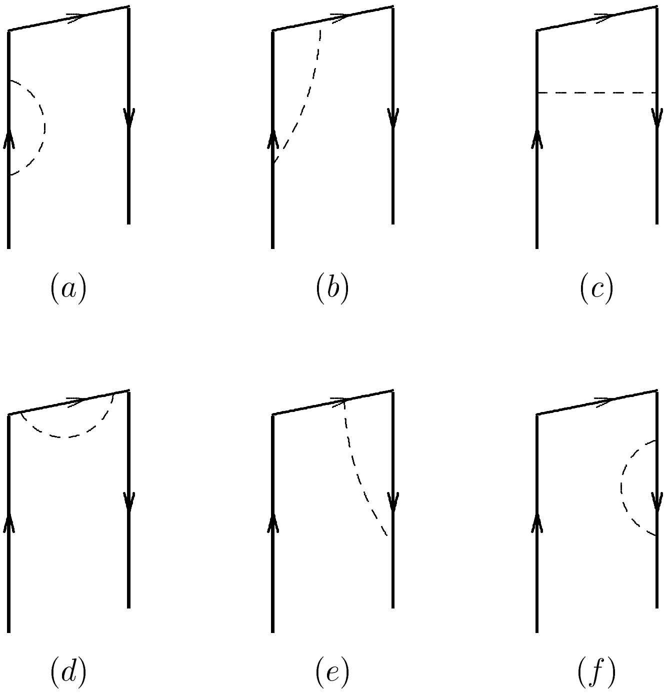


Figure 2: Diagrams of the order λ for the expectation value of the Π-shape Wilson loop. The dashed lines represent either scalar or gauge-field propagators. Only the diagrams in figures (b) and (e) contribute to the anomalous dimension γ_n.

The diagram in figure 2b reads for the regularization (2.19) as

$$W^{(1)}_{(\mathrm{b})} = \frac{\lambda}{4\pi^2} uv \int\limits_0^\infty \mathrm{d}s \int\limits_0^T \mathrm{d}t \frac{1}{s^2 - 2uvst - \epsilon^2}\,, \tag{2.20}$$

where the light-cone segment is parametrized by t and the infinite segment is parametrized by s. Using the variable (2.4), we rewrite eq. (2.20) as

$$\begin{aligned} W^{(1)}_{(\mathrm{b})} &= \frac{\lambda}{4\pi^2} \int\limits_0^\infty \mathrm{d}s \int\limits_0^L \mathrm{d}x \frac{1}{s^2 - 2sx - \epsilon^2}\,, \\ &= -\frac{\lambda}{8\pi^2} \int\limits_0^L \mathrm{d}x \frac{1}{\sqrt{x^2+\epsilon^2}} \log \frac{x + \sqrt{x^2+\epsilon^2}}{x - \sqrt{x^2+\epsilon^2}}\,. \end{aligned} \tag{2.21}$$

Substituting

$$\frac{L}{\epsilon} = -\mathrm{i}n\,, \tag{2.22}$$

as is prescribed by eq. (2.8), we get finally

$$\begin{aligned} W^{(1)}_{(\mathrm{b})} &= -\frac{\lambda}{8\pi^2}\left[\frac{1}{4}\log^2\left(\frac{n+\sqrt{n^2-1}}{n-\sqrt{n^2-1}}\right) + \frac{\pi^2}{4}\right] \\ &\overset{n\to\infty}{=} -\frac{\lambda}{8\pi^2}\left[\log^2(2n) + \frac{\pi^2}{4}\right]. \end{aligned} \tag{2.23}$$

The $\log^2$-term is as in ref. [18].

The diagram in figure 2*e* gives exactly the same result as that in figure 2*b*.

The diagram in figure 2*c* connects the two infinite vertical segments. It contributes to the interaction potential (see e.g. ref. [1], p. 255) but does not have the $\log^2(L/\epsilon)$ which contributes to γ_n. The diagram in figure 2*c* is therefore not essential for the calculation of the anomalous dimension.

Adding the diagrams in figures 2*b* and *e*, we obtain for the anomalous dimension to order λ

$$\gamma_n = \frac{\lambda}{2\pi^2} \log n \tag{2.24}$$

which reproduces the result of an explicit calculation in ref. [24].

It is worth noting that the right-hand side of eq. (2.23) became real only after the analytic continuation (2.22). For real L it has an imaginary part associated with the divergence of the integral over s in eq. (2.21) for $s = x + \sqrt{x^2 + \epsilon^2}$. The meaning of such an imaginary part for the Wilson loops with cusps in Minkowski space is discussed in ref. [25]. This imaginary part will also appear in section 3.

2.4 Remark on cusp near the light cone

As is pointed out in ref. [18], the anomalous dimension γ_n is determined by the cusp anomalous dimension $\Gamma_{\rm cusp}$ associated with the renormalization of Wilson loops with cusps as depicted in figure 3.

The precise relation is as follows:

$$\Gamma_{\rm cusp} \stackrel{\gamma\to\infty}{=} \frac{\gamma}{2} f(\lambda)\,, \tag{2.25}$$

where γ is the cusp angle defined in Minkowski space by

$$\cosh \gamma = \frac{uv}{\sqrt{v^2}} \tag{2.26}$$

and the function $f(\lambda)$ enters eq. (2.9).

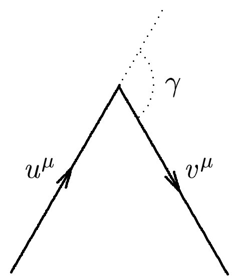


Figure 3: Γ-shape Wilson loop having a cusp. The cusp angle γ is given by eq. (2.26).

It is easy to obtain eq. (2.25) taking the light-cone limit ($v^2 \to 0$) of the renormalization-group equation satisfied by (2.7) and noting that $\gamma \to \infty$ in this limit. The cusp anomalous dimension in the limit of large γ was discussed in ref. [25].

Note that the diagrams in figures 2*b* and *e* look similar to the diagrams which give the cusp anomalous dimension. The difference is that in our case one segment of the Γ-shape loop lies strictly at the light cone, so the term $\gamma \log(L/\epsilon)$ in the vacuum expectation value of the Wilson loop with the cusp is replaced by $(\log^2(L/\epsilon))/2$ in our case. We find it more convenient to deal directly with the light-cone Wilson loop than to approach the light cone first considering the loop in figure 3 at finite γ and then taking the limit $\gamma \to \infty$.

3. Open string in $AdS_5 \times S^5$

As is shown in refs. [6, 7], the supersymmetric Wilson loop (2.12) is dual to an open string in $AdS_5 \times S^5$, the ends of which run along the contour $\{x^\mu(s), |x(s)|\theta^i\}$ in the boundary of $AdS_5 \times S^5$. In the supergravity limit, the string worldsheet coincides with the minimal surface in AdS_5 bounded by $x^\mu(s)$. This determines the asymptotic behavior of the Wilson loop for large λ. While the original solutions, obtained in refs. [6, 7] for an (infinite) rectangle, in refs. [8, 9] for a circle, and in ref. [9] for the loop with a cusp depicted in figure 3, are given in euclidean space, their Minkowski-space analogs are known [26] in the former two cases.

3.1 Finding the minimal surface

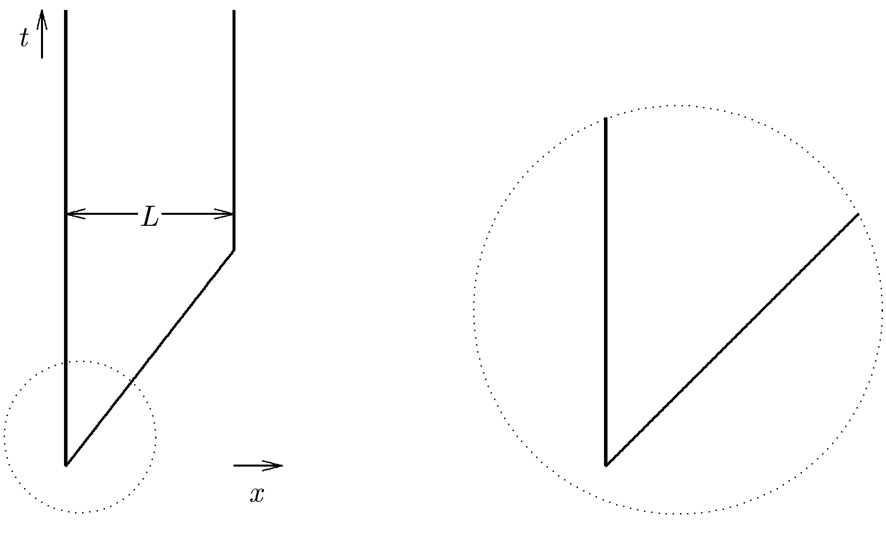
To calculate the minimal surface for the Minkowski-space loop in figure 1, we parametrize the worldsheet by the coordinates $t = x^0$ and $x = x^1$ for the choice (2.3) of the Π-shape contour as is depicted in figure 4a. The Nambu–Goto action of an open string in AdS_3 space with the metric given in the Poincaré coordinates by

$$\mathrm{d}s^2 = R^2 \frac{-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}z)^2}{z^2} \tag{3.1}$$

reads as

$$S = -\frac{R^2}{2\pi\alpha'} \int\limits_0^L \mathrm{d}x \int\limits_x^\infty \mathrm{d}t \frac{1}{z^2} \sqrt{1 + z'^{\,2} - \dot{z}^{\,2}}\,, \tag{3.2}$$

where $\dot{z} = \mathrm{d}z/\mathrm{d}t$ and $z' = \mathrm{d}z/\mathrm{d}x$. We consider the AdS_3 subspace of AdS_5 space because we set $x^2 = x^3 = 0$ for the location of the minimal surface.



(a) (b)

Figure 4: Π-shape loop (a) bounding the minimal surface. The rotated segment lies at the light cone. The $\log^2$-term comes from the region near the cusp which is magnified in (b). There are two such regions associated with the two cusps. The typical size of the magnified region is $\sim L$.

The minimal surface $z(t, x)$ which is obtained by minimizing the action (3.2) depends in general both on t and x. However, the $\log^2$-term yielding the anomalous dimension comes from the part of the minimal surface near the cusp as is shown in figure 4. In this case we substitute

$$z(t, x) = \sqrt{t^2 - x^2}\frac{1}{f\left(\frac{x}{t}\right)} \tag{3.3}$$

so that only a function of the dimensionless ratio x/t is to be determined. This reminds the calculation of the cusp anomalous dimension in ref. [9].

Another important limiting case is when $t \gg x$, where z does not depend on t owing to translational symmetry of the problem. This reproduces the calculation of the interaction potential in refs. [6, 7] and perfectly agrees with what is said at the end of Subsection 2.3 concerning the diagram in figure 2*c* which contributes only to the interaction potential rather than to the cusp anomalous dimension.

Before substituting the ansatz (3.3) into eq. (3.2), it is convenient to parametrize the worldsheet by the coordinates

$$r = x\,, \qquad \theta = \operatorname{arctanh}\frac{x}{t} \tag{3.4}$$

so that $0 < r < L$ and $0 < \theta < \infty$. The two rays of the contour in figure 4*b* correspond to $\theta = 0$ and $\theta = \infty$. Then we rewrite the action (3.2) as

$$S = -2\frac{R^2}{2\pi\alpha'}\int\frac{\mathrm{d}r}{r}\int \mathrm{d}\theta\sqrt{f^4 - f^2 + f'^2}\,, \tag{3.5}$$

where the factor of 2 is because the contour has two cusps. The difference from its euclidean counterpart of ref. [9] is the sign in front of the f^2-term.

The function $f(\theta)$ obeys the boundary condition

$$f(0) = \infty \tag{3.6}$$

for the minimal surface to end up in the boundary of AdS space. There is no boundary condition imposed on $f(\infty)$ since $z = 0$ for the ansatz (3.3) when $t = x$, i.e. when the surface approaches the light cone in the boundary. The light cone, given in AdS space by $z^2 = t^2 - x^2$, corresponds to $f = 1$, while the Poincaré horizon would be associated with $f = 0$ in these coordinates.

In analogy with refs. [6, 9], the extremum of (3.5) is given by the analytic function[2]

$$\theta = \sqrt{2}\operatorname{arctanh}\sqrt{2(1-f^2)} - \operatorname{arctanh}\sqrt{1-f^2} - \frac{\mathrm{i}\pi}{2}\left(\sqrt{2}-1\right). \tag{3.7}$$

This solution obeys (3.6) for $\theta = 0$ and $f \to 1/\sqrt{2}$ as $\theta \to \infty$. When f approaches $1/\sqrt{2}$, the minimal surface approaches the one found in ref. [17] and given by

$$z = \sqrt{2(t^2 - x^2)}\,. \tag{3.8}$$

[2]More details are presented in appendix A.

Noting that $f(\theta)$, given by an inverse to eq. (3.7), satisfies

$$f' = f\sqrt{1-f^2}\left(1-2f^2\right), \tag{3.9}$$

we see that the right-hand side of eq. (3.5) is linearly divergent for $f \to \infty$ and logarithmically divergent for $f \to 1/\sqrt{2}$. After the standard subtraction of the linear divergence, we have the function

$$\begin{aligned} A(f) &= \frac{R^2}{\pi\alpha'}\int \frac{\mathrm{d}r}{r}\left\{\int_f^\infty \mathrm{d}f\left[f\frac{\sqrt{f^2-1}}{f^2-1/2}-1\right]-f\right\} \\ &= \mathrm{i}\frac{R^2}{\pi\alpha'}\int\frac{\mathrm{d}r}{r}\left\{\frac{1}{2\sqrt{2}}\left[\operatorname{arctanh}\left(\frac{\sqrt{1-f^2}}{\sqrt{2}-f}\right)+\operatorname{arctanh}\left(\frac{\sqrt{1-f^2}}{\sqrt{2}+f}\right)\right]-\sqrt{1-f^2}\right\} \end{aligned} \tag{3.10}$$

which gives the area of the part of the minimal surface from $\theta = 0$ to $\theta = \theta(f)$.

For $\theta \to \infty$ when $f \to 1/\sqrt{2}$, the first arctanh in square brackets logarithmically diverges. One more logarithmic divergence comes from the integral of $\mathrm{d}r/r$.

To regularize these divergences, we proceed in the standard way [6] shifting the boundary to $z = \epsilon$. Then the consideration is quite similar to that in ref. [9] for the case of the loop with a cusp. From eqs. (3.3) and (3.4), we have

$$f(\theta)\sinh\theta < \frac{r}{\epsilon}. \tag{3.11}$$

For $\theta \gg 1$ this implies,

$$\theta < \theta_{\max} = \log\frac{2\sqrt{2}r}{\epsilon} \tag{3.12}$$

and from eq. (3.7) we obtain

$$f > f_{\min} = \frac{1}{\sqrt{2}} + \sqrt{2}\,\mathrm{e}^{-\sqrt{2}\theta_{\max}}. \tag{3.13}$$

Also r is bounded above by the value of the order of L which is the size of the magnified region in figure 4.

Evaluating the right-hand side of eq. (3.10) for $f = f_{\min}$, we finally obtain the $\log^2$-term

$$A(f_{\min}) = \mathrm{i}\frac{R^2}{2\pi\alpha'}\int\frac{\mathrm{d}r}{r}\,\theta_{\max}(r) = \mathrm{i}\frac{R^2}{4\pi\alpha'}\log^2\frac{L}{\epsilon}. \tag{3.14}$$

It is easy to obtain the same answer directly from eq. (3.5) substituting $f = 1/\sqrt{2}$ for large θ which results in the term linear in $\theta_{\max}$. The latter procedure is equivalent to saying that the area of the minimal surface is dominated by the contribution from its part described by eq. (3.8) which was also the case in ref. [17]. This is the reason why our results for the anomalous dimension agree, while the solutions for the minimal surface are different.

The appearance of the factor of i in eq. (3.14) is remarkable since then $\mathrm{i}A(f_{\min})$, the exponential of which is to be compared with W_{A} on the SYM side

$$W_{\mathrm{A}}(\Pi) = \mathrm{e}^{2\mathrm{i}A(f_{\min})}, \tag{3.15}$$

is real. Here the factor of 2 in the exponent is owing to eq. (2.15). Differentiating with respect to ϵ according to eqs. (2.7) and (2.8), we obtain

$$\gamma_n = \frac{R^2}{\pi\alpha'}\log n \tag{3.16}$$

which reproduces eqs. (1.1) and (1.2) for $R^2/\alpha' = \sqrt{\lambda}$.

Some comments concerning the solution (3.7) are in order.

The inverse function $f(\theta)$ is complex-valued for real $0 < \theta < \infty$. Analogously, the $\log^2(L/\epsilon)$-term in $A(f_{\min})$ given by eq. (3.14) is pure imaginary, while the $A(f)$ itself is complex in general. This property of $A(f_{\min})$ agrees with what we had already discussed at the end of Subsection 2.3 for the light-cone Wilson loops in SYM. There is nothing wrong, in principle, having such a complex saddle-point trajectory in the path integral. Its possible quantum-mechanical interpretation will be discussed in Subsection 3.2.

If we were to use the variables r and f instead of r and θ to parametrize the worldsheet, then θ given by eq. (3.7) would be pure imaginary for $f > 1$. Such an imaginary

$$\theta = -\mathrm{i}\tau \tag{3.17}$$

could be viewed as if x is pure imaginary or, in other words, we have performed the analytic continuation (2.22) after which W was expected to become real. The fact that $A(f)$ is real for $f > 1$, which corresponds to $\tau < \pi(\sqrt{2}-1)/2$, does not contradict the expected behavior of $W(\Pi)$ in SYM. When f becomes smaller than 1 to approach the value $f_{\min}$, then τ and $A(f)$ become complex. One should keep in mind that eq. (3.10) is only an approximation of the area of the whole minimal surface, spanned by the loop in figure 4a, by its parts near the cusps, which was used only to evaluate the $\log^2(L/\epsilon)$. This minimal surface is, as has been already mentioned, a nontrivial function of two variables and is given by the extremum of the action (3.2) rather than (3.5).

The value $f = 1$ is associated with the light cone in AdS space. Therefore, the values $f < 1$ mean that one is no longer inside the light cone. We discuss an appropriate quantum-mechanical interpretation of this situation in the next subsection.

3.2 Quantum-mechanical interpretation

After the analytic continuation (3.17) we are dealing with a saddle-point trajectory of the quantum-mechanical problem with the time-variable τ, while the right-hand side of eq. (3.15) is related to the phase a semiclassical (Schrödinger) wave function by

$$\log\Psi(f) = \mathrm{i}\frac{2R^2}{\pi\alpha'}\int\frac{\mathrm{d}r}{r}\left\{\int_f^\infty \mathrm{d}f\left[\frac{f^2-1/2}{f\sqrt{f^2-1}}-1\right]-f\right\}. \tag{3.18}$$

We obtain from eq. (3.7)

$$\tau = -\sqrt{2}\arctan\sqrt{2(f^2-1)} + \arctan\sqrt{f^2-1} + \frac{\pi}{2}\left(\sqrt{2}-1\right). \tag{3.19}$$

The inverse function $f(\tau)$ is now real for $\tau \le \pi(\sqrt{2}-1)/2$ which means $f \ge 1$. Alternatively, f becomes complex for $\tau > \pi(\sqrt{2}-1)/2$.

As is already mentioned, $A(f)$ is real for $f > 1$ and complex for $f < 1$. The wave function in eq. (3.18) is therefore oscillating for $f > 1$ and exponentially damped for $f < 1$.[3] This is a typical behavior for a quantum-mechanical problem with the potential that forbids for a classical particle to penetrate the $f < 1$ region. More arguments in favor of such an interpretation are presented in appendix A, where it is argued that the effective potential has an infinite wall at $f = 1$: it approaches $-\infty$ when f tends to 1 from above and then decreases from $+\infty$.

The region $f < 1$ can be penetrated quantum-mechanically by tunneling under the barrier, thus approaching the point $f = f_{\min} \approx 1/\sqrt{2}$ that determines the anomalous dimension. The wave function is then exponentially dumped as is displayed in eqs. (3.14) and (3.18). Note that $\theta_{\max}$ cancels on the right-hand side of eq. (3.18) which is regular as $f \to 1/\sqrt{2}$.

Once again, this quantum-mechanical interpretation of the minimal surface which determines the anomalous dimension is linked to the mechanical analogy we are considering.

4. Conclusion

The light-cone Wilson loops are convenient for calculating the anomalous dimensions of conformal operators of twist two with large Lorentz spin both on the SYM and AdS sides. The lowest-order perturbation-theory calculation in SYM is in a qualitative agreement with the result given by the area of the minimal surface formed by the worldsheet of an open string ending at the loop in the boundary of AdS space. The latter coincides in turn with the one [15] obtained from the classical closed folded string rotating in AdS_5.

It would be interesting to pursue the calculations in both cases to next orders to compare the results. There is no problem to calculate the anomalous dimensions to order λ^2 in SYM, which is quite similar to the calculation of ref. [18] for QCD. The result is to be compared with an explicit calculation of ref. [27] to this order. Analyzing the simplest rainbow graph to the order λ^2, we immediately find as in refs. [25, 18] that the sum of the diagrams with the triple vertex (either of three gauge fields or one gauge field and two scalars) has to have a term

$$W^{(2)} \sim \lambda^2 \log^4 \frac{L}{\epsilon} \tag{4.1}$$

which is required for the exponentiation of the result (2.24) from the previous order λ. In contrast to the circular Wilson loop in ref. [10], the diagrams with interactions are now present. Perhaps, they can still be analyzed similarly to ref. [11] making a 2D conformal transformation in the boundary, which maps the straight line (2.13) into a loop of the type in figure 4*b*. Since it is not a symmetry of AdS space, this would rewrite its metric in new coordinates.

[3] There could be an additional difference of phases of the wave function in the two regions, which is calculable either semiclassically or analyzing the region $f \approx 1$.

The structure of the calculation of the anomalous dimension via the minimal surface on the AdS side suggests in turn that it might be completely determined by a certain state (or states) in the (quantum) spectrum of the open string in the AdS background.

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A. Solving the classical equation

After the analytic continuation (3.17) the action (3.5) takes the form

$$S = \frac{R^2}{\pi\alpha'}\int \frac{\mathrm{d}r}{r}\int \mathrm{d}\tau \sqrt{\dot{f}^2 - f^4 + f^2}\,. \tag{A.1}$$

Note that the action (A.1) remains real, so the analytic continuation (3.17) is not the same as introducing an imaginary time for a system with a hamiltonian which is quadratic in momentum.

The minimal surface is described by the Euler–Lagrange equation

$$\ddot{f}V = \dot{f}^2 V' - \frac{1}{2}V'V \qquad \text{with} \quad V = f^4 - f^2\,, \tag{A.2}$$

where $\dot{f} = \mathrm{d}f/\mathrm{d}\tau$ and $V' = \mathrm{d}V/\mathrm{d}f$, which is associated with the lagrangian in eq. (A.1) for $f^4 - f^2$ substituted by V. It is to be solved on the interval $[0,\infty)$ with the boundary condition (3.6).

Equation (A.2) is analogous to the equation of motion for a mechanical system with a velocity-dependent force. The conserved "energy" is given by

$$E = \frac{V}{\sqrt{\dot{f}^2 - V}} \tag{A.3}$$

which for $E = 1/2$ determines

$$\dot{f} = -\sqrt{V + \frac{V^2}{E^2}} = -\frac{f\sqrt{f^2-1}}{f^2 - 1/2}\,. \tag{A.4}$$

This reproduces the solution (3.19). The "turning" point of this trajectory is at $f = 1/\sqrt{2}$. Such a special solution is needed to have a trajectory which gives the $\log^2 L/\epsilon$ in eq. (3.14).

The value of the momentum along the trajectory is given by

$$p = \frac{\dot{f}}{\sqrt{\dot{f}^2 - V}} = -\sqrt{1 + \frac{E^2}{V}} = -\frac{f^2 - 1/2}{f\sqrt{f^2-1}} \tag{A.5}$$

which results in eq. (3.18). We see that p is real for $f > 1$ and becomes infinite as $f \to 1$. Then it is imaginary for $f < 1$. This looks like the effective potential has an infinite wall at $f = 1$: approaching $-\infty$ for $f \to 1$ from above and then decreasing from $+\infty$. Therefore, classical motion is allowed only for $f > 1$ (see also appendix B).

One can penetrate, however, the region $f < 1$ quantum-mechanically approaching the point $f = f_{\min} \approx 1/\sqrt{2}$ by tunneling under the barrier. The wave function is then exponentially dumped as is shown in eqs. (3.14) and (3.18).

The region near $f = 1$ can be directly analyzed substituting

$$f(\tau) = 1 + \xi\, \mathrm{e}^{g(\tau/\sqrt{\xi})} \tag{A.6}$$

with $\xi \ll 1$. Equation (A.2) then takes the form

$$\ddot{g} = -\,\mathrm{e}^{-g} \tag{A.7}$$

which coincides with the classical equation of motion for the lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{g}^2 + \mathrm{e}^{-g}\,. \tag{A.8}$$

The potential of this problem is $V = -\,\mathrm{e}^{-g}$ which exponentially falls down as $g \to -\infty$, i.e. $f \to 1$ from above. This is the same assertion as above.

B. Relation to time-like geodesics

An alternative point of view on the classical problem of appendix A is as that of constructing geodesics in space with the metric

$$\mathrm{d}s^2 = \mathrm{d}f^2 - V(f)\,\mathrm{d}\tau^2\,. \tag{B.1}$$

The equations for time-like geodesics can then be derived from the lagrangian which is quadratic in the derivatives of τ and f with respect to the proper time s and read as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(V(f)\frac{\mathrm{d}\tau}{\mathrm{d}s}\right) = 0\,, \tag{B.2}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + V'(f)\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 = 0 \tag{B.3}$$

with the constraint

$$\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 - V(f)\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 = 1\,. \tag{B.4}$$

Solving the constraint for $(\mathrm{d}\tau/\mathrm{d}s)^2$ and substituting into eq. (B.3), we obtain the equation

$$V\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + V'\left[\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 - 1\right] = 0 \tag{B.5}$$

which determines the geodesic $f(s)$.

A convenient way to solve eq. (B.5) is as follows. From eq. (B.2) we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{E}{V}\,, \tag{B.6}$$

where E is an integration constant. Substituting in eq. (B.4), we find

$$\frac{\mathrm{d}f}{\mathrm{d}s} = -\sqrt{1 + \frac{E^2}{V}} \tag{B.7}$$

which yields

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = -\sqrt{V + \frac{V^2}{E^2}}\,. \tag{B.8}$$

Equations (B.7) and (B.8) are the same as eqs. (A.5) and (A.4) for $V = f^4 - f^2$ and $E = 1/2$. The conclusion is the same: these geodesics also exist only for $f > 1$ where the signature of the metric (B.1) is lorentzian.

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